JOURNAL OF APPROXIMATION THEORY 38, 81–89 (1983)

Quantitative Estimates for L_p Approximation with Positive Linear Operators

J. J. SWETITS

Department of Mathematical Sciences, Old Dominion University, Norfolk, Virginia 23508, U.S.A.

AND

B. WOOD

Mathematics Department, University of Arizona, Tucson, Arizona 85721, U.S.A.

Communicated by E. W. Cheney

Received December 17, 1981

Quantitative estimates for approximation with positive linear operators are derived. The results are in the same vein as recent results of Berens and DeVore. Two examples are provided.

1. INTRODUCTION

Berens and DeVore [1, 2] have recently obtained quantitative estimates for L_p approximation with positive linear operators. The results may be formulated as follows: Let $I=[a,b]$ and let $L_p(I)$ $(1 \leqslant p < \infty)$ denote the space of measurable real-valued pth power Lebesgue integrable functions f on I with $\|f\|_p = (\int_a^b |f|^p)^{1/p}$. For $f \in L_p(I)$, define the second-order integral modulus of smoothness as

$$\omega_{2,p}(f,h) = \sup_{0<t\leqslant h} \|f(\cdot+t) - 2f(\cdot) + f(\cdot - t)\|_{L_p(I_{2t})},$$

where $L_p(I_{2t})$ indicates that the L_p norm is taken over $[a+t, b-t]$. Let $e_i(t)=t^i$ for $i=0,1,2$. A linear map L from $L_p(I)$ into $L_p(I)$ is called a contraction if $\|L(f)\|_p \leqslant \|f\|_p$ for all $f \in L_p(I)$. Let $\{L_n\}$ be a sequence of positive linear operators from $L_p[a,b]$ into $L_p[c,d]$, $a \leqslant c < d \leqslant b$, and define

$$\lambda_{np} = (\max_{i=0,1} \|L_n(e_i) - e_i\|_{L_p[c,d]})^{1/2},$$

0021-9045/83 $3.00

and

$$\hat{\lambda}_{np} = (\max_{i=0,1,2} \|L_n(e_i) - e_i\|_{L_p[c,d]})^{1/2}.$$

In [1] it is shown that if $\{L_n\}$ is a sequence of positive linear contractions from $L_p[a,b]$ into $L_p[a,b]$, then, for $f \in L_p[a,b]$,

$$\|L_n(f) - f\|_{L_p[a,b]} \leqslant C_1[\lambda_{np}^{2/p} \|f\|_p + \omega_{2,p}(f, \lambda_{np}^{1/p})], \tag{1.1}$$

where $C_1 > 0$ is independent of f and n.

In [2] it is shown that if $\{L_n\}$ is a sequence of positive linear operators from $L_p[a,b]$ into $L_p[c,d]$, then, for $f \in L_p[a,b]$,

$$\|L_n(f) - f\|_{L_p[c,d]} \leqslant C_2[\hat{\lambda}_{np}^{4p/(2p+1)} \|f\|_p + \omega_{2,p}(f, \hat{\lambda}_{np}^{2p/(2p+1)})], \tag{1.2}$$

where $C_2 > 0$ is independent of f and n. Estimate (1.2) is good for large p, while (1.1) is effective for positive linear contractions with p close to 1. In general, (1.1) and (1.2) cannot be improved, and (1.1) is not valid for contraction operators that map $L_p[a,b]$ into $L_p[c,d]$, $a < c < d < b$.

Many well-known sequences of positive linear operators have a rate of convergence that is better than that predicted by (1.1) and (1.2) (see, e.g., [1, 7, 10]). These sequences satisfy the estimate

$$\|L_n(f) - f\|_{L_p[c,d]} \leqslant M_p[\hat{\lambda}_{np}^2 \|f\|_p + \omega_{2,p}(f, \hat{\lambda}_{np})], \tag{1.3}$$

where $M_p > 0$ is independent of f and n.

The estimate (1.3) is the L_p analog of Freud's optimal estimate [4] for approximation in the space $C[a,b]$.

The purpose of this paper is to investigate conditions under which (1.3) can be attained. Specifically, let

$$\mu_{np} = (\max\{\|L_n(e_0) - e_0\|_{L_p[c,d]}, \\ \|L_n((t-x), x)\|_{L_p[c,d]}, \|L_n((t-x)^2, x)\|_{L_p[c,d]}^{2p/(2p+1)}\})^{1/2},$$

where $\{L_n\}$ is a sequence of positive linear operators from $L_p[a,b]$ into $L_p[c,d]$. We prove

THEOREM 1. *Let $\{L_n\}$ be uniformly bounded sequence of positive linear operators from $L_p[a,b]$ into $L_p[c,d]$, $1 \leqslant p < \infty$, $a \leqslant c < d \leqslant b$, and assume $\mu_{np} \to 0$ $(n \to \infty)$. Then for $f \in L_p[a,b]$ and n sufficiently large,*

$$\|L_n(f) - f\|_{L_p[c,d]} \leqslant C_p(\mu_{np}^2 \|f\|_p + \omega_{2,p}(f, \mu_{np})), \tag{1.4}$$

where $C_p > 0$ is independent of f and n.

Estimate (1.4) is never worse than (1.2). If $\mu_{np}^2 = \max\{\|L_n(e_0)-e_0\|_{L_p[c,d]}, \|L_n((t-x),x)\|_{L_p[c,d]}\}$, then estimate (1.4) is better than (1.1) (for $p>1$) or (1.2), and it is equivalent to (1.3). This is the case for the convolution operators of [10, Remarks, p. 362 and Lemma 1, p. 356].

The second result requires some additional information about the approximation properties of $\{L_n\}$. Here we deal with sequences $\{L_n\}$ such that

$$\mu_n \equiv \|L_n((t-x)^2, x)\|_{L_\infty[c,d]} \tag{1.5}$$

exists for each n and

$$t_{np} = (\max\{\|L_n(e_0)-e_0\|_{L_p[c,d]}, \|L_n((t-x),x)\|_{L_p[c,d]}, \mu_n\})^{1/2}.$$

THEOREM 2. *Let $\{L_n\}$ be a uniformly bounded sequence of positive linear operators from $L_p[a,b]$ into $L_p[c,d]$ such that $t_{np}\to 0$ $(n\to\infty)$.*

(i) *If $p>1$ and $f\in L_p[a,b]$, then*

$$\|L_n(f)-f\|_{L_p[c,d]} \leqslant M_p[t_{np}^2\|f\|_p + \omega_{2,p}(f,t_{np})],$$

where $M_p>0$ is independent of f and n.

(ii) *If there exists $\alpha>3$ such that*

$$\|L_n(|t-x|^\alpha, x)\|_{L_\infty[c,d]} = O(\mu_n^{\alpha/2}) \qquad (n\to\infty),$$

then, for $f\in L_1[a,b]$,

$$\|f-L_n(f)\|_{L_1[c,d]} \leqslant M_1[t_{n1}^2\|f\|_1 + \omega_{2,1}(f,t_{n1})],$$

where $M_1>0$ is independent of f and n.

The estimates of Theorem 2 are equivalent to (1.3) and are better than the Berens–DeVore estimates when

$$\mu_n = O(\|L_n((t-x)^2,x)\|_{L_1[c,d]}) \qquad (n\to\infty). \tag{1.6}$$

This is the case for the Bernstein–Kantorovic operators of [1, 3, 7]. However, even if (1.6) is not satisfied, the following example shows that the estimates of Theorem 2 can be sharp in cases where those of Theorem 1 are not;

Fix $\alpha>0$, $\beta>0$, and, for n sufficiently large, define $L_n: L_p[0,1]\to L_p[0,1]$ by

$$\begin{aligned} L_n(f(t),x) &= f(x) & |x-\tfrac12| &\geqslant n^{-\beta}, \\ &= \frac{n^\alpha}{2}\int_{n^{-\alpha}}^{n^\alpha} f(t+x)\,dt, & |x-\tfrac12| &< n^{-\beta}. \end{aligned}$$

Then $L_n(e_i(t), x) = e_i(x)$, $i = 0, 1$, and L_n is not a contraction mapping. Additionally, we have

$$\begin{aligned} L_n((t-x)^2, x) &= 0, & |x - \tfrac{1}{2}| \geqslant n^{-\beta}, \\ &= n^{-2\alpha}/3, & |x - \tfrac{1}{2}| < n^{-\beta}, \end{aligned}$$

and

$$\sup_{0 \leqslant x \leqslant 1} L_n((t-x)^4, x) = n^{-4\alpha}/5.$$

Theorem 1 yields

$$\begin{aligned} \|L_n(f) - f\|_{L_p[0,1]} \leqslant C_p(\|f\|_p n^{(-2\alpha - (\beta/p))(2p/(2p+1))} \\ + \omega_{2,p}(f, n^{(-2\alpha - (\beta/p))(p/(2p+1))})), \end{aligned}$$

while Theorem 2 yields

$$\|L_n(f) - f\|_{L_p[0,1]} \leqslant C_p(\|f\|_p n^{-2\alpha} + \omega_{2,p}(f, n^{-\alpha})).$$

The latter estimate is better than the former if $\alpha > \beta$, since $n^{-\alpha} = o(n^{(-2\alpha - (\beta/p))p/(2p+1)})$ $(n \to \infty)$ in this case.

Assuming $\alpha > \beta$, then straightforward calculations establish the existence of constant $k_p > 0$ independent of n such that

$$\|L_n((t - \tfrac{1}{2})_+, x) - (x - \tfrac{1}{2})_+\|_{L_p[0,1]} \geqslant k_p n^{-\alpha(1 + 1/p)},$$

where

$$\begin{aligned} (t - \tfrac{1}{2})_+ &= 0, & 0 \leqslant t \leqslant \tfrac{1}{2}, \\ &= t - \tfrac{1}{2}, & \tfrac{1}{2} \leqslant t \leqslant 1. \end{aligned}$$

Since $\omega_{2,p}((t - \frac{1}{2})_+, \delta) = O(\delta^{1+1/p})$ $(\delta \to 0^+)$, this shows that the estimate of Theorem 2 is sharp.

2. Proofs of the Theorems

Let $L_p^{(2)}(I)$ be the space of those functions $f \in L_p(I)$ with f' absolutely continuous and $f'' \in L_p(I)$.

The keys to the proofs of Theorems 1 and 2 are the following lemmas:

LEMMA 1. *Assume the hypotheses of Theorem 1 are satisfied. For all n sufficiently large and for $f \in L_p^{(2)}[a, b]$,*

$$\|L_n(f) - f\|_{L_p[c,d]} \leqslant C'_p(\|f\|_p + \|f''\|_p)\mu_{np}^2.$$

Proof. Let $f \in L_p^{(2)}[a, b]$ and assume f has been extended outside of $[a, b]$ so that $f''(x) = 0$ if $x \notin [a, b]$.

For $t \in [a, b]$ and $x \in [c, d]$, we have

$$f(t) - f(x) = f'(x)(t - x) + \int_x^t (t - u) f''(u)\, du.$$

Thus

$$\begin{aligned} \|L_n(f(t) - f(x), x)\|_{L_p[c,d]} &\leqslant \|f'\|_{L_\infty[c,d]} \|L_n(t - x, x)\|_{L_p[c,d]} \\ &\quad + \left\| L_n \left(\int_x^t (t - u) f''(u)\, du, x \right) \right\|_{L_p[c,d]}. \end{aligned}$$

Fix $\delta > 0$. If $|t - x| \leqslant \delta$, then

$$\left| \int_x^t (t - u) f''(u)\, du \right| \leqslant \delta \int_0^\delta |f''(x + u)|\, du.$$

If $|t - x| > \delta$, then, using Hölder's inequality,

$$\left| \int_x^t (t - u) f''(u)\, du \right| \leqslant |t - x|^{2 - 1/p} \|f''\|_p \leqslant \frac{(t - x)^2}{\delta^{1/p}} \|f''\|_p.$$

Thus

$$\begin{aligned} &\left\| L_n \left(\int_x^t (t - u) f''(u)\, du, x \right) \right\|_{L_p[c,d]} \\ &\quad \leqslant \left\| \delta L_n(e_0, x) \int_0^\delta |f''(x + u)|\, du \right\|_{L_p[c,d]} \\ &\quad\quad + (\|f''\|_p / \delta^{1/p}) \|L_n((t - x)^2, x)\|_{L_p[c,d]}. \end{aligned}$$

The first term is dominated by

$$\begin{aligned} &\delta \left\| (L_n(e_0, x) - e_0) \int_0^\delta |f''(x + u)|\, du \right\|_{L_p[c,d]} \\ &\quad + \delta \left\| \int_0^\delta |f''(x + u)|\, du \right\|_{L_p[c,d]} \\ &\leqslant \delta^{2 - 1/p} \|f''\|_p \|L_n(e_0) - e_0\|_{L_p[c,d]} + \delta \int_0^\delta \|f''(x + u)\|_p\, du \\ &\leqslant \|f''\|_p (\delta^{2 - 1/p} \|L_n(e_0) - e_0\|_{L_p[c,d]} + \delta^2), \end{aligned}$$

where the latter inequalities follow from Hölder's inequality, the generalized Minkowski inequality [9, p. 592], and the fact that $f''(x)=0$ if $x \notin [a,b]$.

Therefore

$$\left\| L_n\left(\int_x^t (t-u)\, f''(u)\, du, x\right)\right\|_{L_p[c,d]}$$
$$\leqslant \|f''\|_p(\delta^{2-1/p}\, \|L_n(e_0)-e_0\|_{L_p[c,d]} + \delta^2 + \frac{1}{\delta^{1/p}}\, \|L_n((t-x)^2, x)\|_{L_p[c,d]}).$$

Since $\|L_n((t-x)^2, x)\|_{L_p[c,d]} \to 0$ $(n \to \infty)$, we can choose, for n sufficiently large,

$$\delta = \|L_n(t-x)^2, x)\|_{L_p[c,d]}^{p/(2p+1)}$$

to obtain

$$\left\| L_n\left(\int_x^t (t-u)\, f''(u)\, du, x\right)\right\|_{L_p[c,d]} \leqslant 3\, \|f''\|_p q_{np},$$

where

$$q_{np} = \max\{\|L_n(e_0)-e_0\|_{L_p[c,d]}, \|L_n((t-x)^2, x)\|_{L_p[c,d]}^{2p/(2p+1)}\}.$$

Thus, we have

$$\|L_n(f)-f\|_{L_p[c,d]} \leqslant \|f\|_{L_\infty[c,d]}\, \|L_n(e_0)-e_0\|_{L_p[c,d]}$$
$$+ \|f'\|_{L_\infty[c,d]}\, \|L_n((t-x), x)\|_{L_p[c,d]} + 3\, \|f''\|_q q_{np}.$$

Using [5, Theorem 3.1], we obtain

$$\|L_n(f)-f\|_{L_p[c,d]} \leqslant C'_p(\|f\|_p + \|f''\|_p)\, \mu_{np}^2.$$

This completes the proof of Lemma 1.

LEMMA 2. *Assume the hypotheses of Theorem 2 are satisfied. Then, for $f \in L_p^{(2)}[a,b]$ and n sufficiently large,*

$$\|L_n(f)-f\|_{L_p[c,d]} \leqslant M'_p(\|f\|_p + \|f''\|_p)\, t_{np}^2,$$

where $M'_p > 0$ is independent of f and n.

Proof. (i) Assume $p>1$ and $f \in L_p^{(2)}[a,b]$. Proceeding as in the proof of Lemma 1, we have

$$\|L_n(f(t)-f(x), x)\|_{L_p[c,d]} \leqslant \|f'\|_{L_\infty[c,d]}\, \|L_n((t-x), x)\|_{L_p[c,d]}$$
$$+ \left\| L_n\left(\int_x^t (t-u) f''(u)\, du, x\right)\right\|_{L_p[c,d]}.$$

The last term is dominated by $\mu_n \|\theta(f'', x)\|_p$, where $\theta(f'', x)$ is the Hardy–Littlewood majorant of f'' at x. As is well known,

$$\|\theta(f'', x)\|_p \leqslant K_p \|f''\|_p,$$

where $K_p > 0$ depends only on p.

Thus,

$$\begin{aligned}\|L_n(f) - f\|_{L_p[c,d]} &\leqslant \|f\|_{L_\infty[c,d]} \|L_n(e_0) - e_0\|_{L_p[c,d]} \\ &+ \|f'\|_{L_\infty[c,d]} \|L_n((t-x), x)\|_{L_p[c,d]} + K_p \mu_n \|f''\|_p.\end{aligned}$$

The rest of the proof follows as in Lemma 1.

(ii) Assume $p = 1$ and $f \in L_p^{(2)}[a, b]$. Assume f has been extended outside of $[a, b]$ so that $f''(x) = 0$ if $x \notin [a, b]$. Then, for $x \in [c, d]$ and $\delta > 0$,

$$\begin{aligned}&\left| L_n \left(\int_x^t (t-u) f''(u)\, du, x \right) \right| \\ &\leqslant \sum_{\substack{j=0 \\ j\delta \leqslant |t-x| < (j+1)\delta}}^{[(b-a)/\delta]} L_n \left(|t-x| \int_0^{(j+1)\delta} |f''(x+u)|\, du, x \right) \\ &\leqslant \left(\delta \int_0^\delta |f''(x+u)|\, du \right) L_n(e_0, x) \\ &\quad + \sum_{j=1}^{[(b-a)/\delta]} \frac{1}{(j\delta)^{\alpha-1}} \int_0^{(j+1)\delta} |f''(x+u)|\, du \cdot L_n(|t-x|^\alpha, x).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\| L_n \left(\int_x^t (t-u) f''(u)\, du, x \right) \right\|_{L_1[c,d]} \\ &\leqslant \delta \|f''\|_1 \|L_n(e_0 - e_0)\|_{L_1[c,d]} + \delta^2 \|f''\|_1 \\ &\quad + \|L_n(|t-x|^\alpha, x)\|_{L_\infty[c,d]} \|f''\|_1 \sum_{j=1}^{\infty} \frac{(j+1)\delta}{(j\delta)^{\alpha-1}},\end{aligned}$$

where the infinite series converges since $\alpha > 3$, and where we have used the fact that $f''(x) = 0$ if $x \notin [a, b]$. Choose $\delta = \mu_n^{1/2}$ to obtain

$$\left\| L_n \left(\int_x^t (t-u) f''(u)\, du, x \right) \right\|_{L_1[c,d]} \leqslant M \|f''\|_1 q_{n1},$$

where $M > 0$ is an absolute constant and

$$q_{n1} = \max\{\|L_n(e_0) - e_0\|_{L_1[c,d]}, \mu_n\}.$$

The rest of the proof follows as in part (i). This completes the proof of Lemma 2.

For $f \in L_p[a, b]$, $1 \leqslant p < \infty$, and $t > 0$, define

$$K_{2,p}(f, t) = \inf_{g \in L_p^{(2)}[a,b]} \{\|f - g\|_p + t(\|g\|_p + \|g''\|_p)\} \tag{2.1}$$

and

$$K'_{2,p}(f, t) = \inf_{g \in L_p^{(2)}[a,b]} \{\|f - g\|_p + t\|g''\|_p\}. \tag{2.2}$$

These are the K-functionals of Peetre [8]. It is known [6] that there are constants s_i, $i = 1, 2, 3, 4$, independent of f and p, such that if $f \in L_p[a, b]$, then

$$s_1 \omega_{2,p}(f, t) \leqslant K_{2,p}(f, t^2) \leqslant \min(1, t^2) \|f\|_p + s_2 \omega_{2,p}(f, t) \tag{2.3}$$

and

$$S_3 \omega_{2,p}(f, t) \leqslant K'_{2,p}(f, t^2) \leqslant s_4 \omega_{2,p}(f, t). \tag{2.4}$$

Proof of Theorem 1. Let $f \in L_p[a, b]$ and $g \in L_p^{(2)}[a, b]$. Since $\{L_n\}$ is a uniformly bounded sequence of positive linear operators from $L_p[a, b]$ into $L_p[c, d]$, we have, by Lemma 1,

$$\|L_n(f) - f\|_{L_p[c,d]} \leqslant (1 + R_p) \|f - g\|_p + C'_p \mu_{np}^2 (\|g\|_p + \|g''\|_p)$$

for all n sufficiently large, where $R_p > 0$ is a uniform bound for $\{L_n\}$. Take the infimum over all $g \in L_p^{(2)}[a, b]$ and use (2.1) and (2.3) to obtain (1.4).

Proof of Theorem 2. The proof is the same as the proof of Theorem 1, except that Lemma 2 is used.

References

1. H. Berens and R. A. DeVore, Quantitative Korovkin theorems for L_p-spaces, *in* "Approximation Theory II" (G. G. Lorentz, Ed.), pp. 289–298, Academic Press, New York, 1976.
2. H. Berens and R. A. DeVore, Quantitative Korovkin theorems for positive linear operators in L_p space, *Trans. Amer. Math. Soc.* **245** (1978), 349–361.
3. Z. Ditzian and C. P. May, L_p saturation and inverse theorems for modified Bernstein polynomials, *Indiana Univ. Math. J.* **25** (1976), 733–751.
4. G. Freud, On approximation by positive linear methods, I and II, *Studia Sci. Math. Hung.* **2** (1967), 63–66, **3** (1968), 365–370.
5. S. Goldberg and A. Meir, Minimum moduli of ordinary differential operators, *Proc. London Math. Soc.* (3) **23** (1971), 1–15.

6. H. JOHNEN, Inequalities connected with the moduli of smoothness, *Math. Vesnik* **9** (24) (1972), 289–303.
7. M. MÜLLER, Die Güte der L_p-Approximation durch Kantorovic-polynome, *Math. Z.* **151** (1976), 243–247.
8. J. PEETRE, "A Theory of Interpolation of Normed Spaces", Lecture Notes, Brazilia, 1963.
9. A. TIMAN, "Theory of Approximation of Functions of a Real Variable," Macmillan Co., N. Y., 1963.
10. B. WOOD, Degree of L_p-approximation with certain positive convolution operators, *J. Approx. Theory* **23** (4) (1978), 354–363.